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LETTER TO THE EDITOR

A Nambu representation of incompressible hydrodynamics using helicity and enstrophy

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Abstract. Nambu mechanics generalizes discrete classical Hamiltonian dynamics. Using the Euler equations for a rotating rigid body, the connection between this representation and non-canonical Hamiltonian mechanics is shown. Nambu mechanics is extended to incompressible ideal hydrodynamical fields using energy and helicity in 3D (enstrophy in 2D). The Hamiltonian and the Casimir invariants of the non-canonical Hamiltonian theory determine the dynamics in a non-singular trilinear bracket.

Nambu (1973) proposed a general representation of classical systems obeying Liouville's theorem. Nambu's mechanics is characterized by the appearance of several conserved quantitites and by the possibility of representing systems with an odd number N of degrees of freedom. In the most simple case, $N = 2$, this reduces to canonical Hamiltonian dynamics. The relation for general N has been investigated and in 1978 Kálnay and Tascón mentioned that '... Nambu mechanics remains independent from the previously known mechanics'. Unfortunately, very few physical systems are known which can be brought in Nambu form. The most well known are the Euler equations of the rotating rigid body (Nambu 1973).

In the last decade, non-canonical Hamiltonian dynamics has been extensively used to investigate non-dissipative hydrodynamics (for adiabatic compressible fluids see Morrison and Greene (1980), and for incompressible fluids, see Olver (1982)). Applications pertain to nonlinear stability analysis (Arnold 1969, Holm *et al* 1985), symmetries (Olver 1982), and approximation theory (geophysical fluid dynamics is reviewed by Salmon (1988), see also Névir (1993)). This theory, which equally applies to systems with an odd number of degrees of freedom, involves an antisymmetric Poisson tensor which may depend on the dynamical variables. In case this tensor is singular, additional conserved quantities denoted as distinguished or Casimir invariants act as constraints on the dynamics in the state space.

The non-canonical Hamiltonian theory of perfect fluid dynamics (Olver 1982) for the Eulerian variables is characterized by the existence of Casimir functionals like helicity (3D) and enstrophy (2D). Motivated by Nambu's approach, we propose an extension of his theory to hydrodynamical fields in which these Casimir functionals appear in a trilinear antisymmetric bracket on the same level as the Hamiltonian.

The Euler equations of a free rigid body mentioned above are quoted frequently as an example for a non-canonical Hamiltonian system (Holm *et al* 1985). With the components of the angular momentum around the principal axes L_i, $i = 1, \ldots, 3$ and the moments of inertia I_i, these equations read as

$$\dot{L}_1 = \frac{I_2 - I_3}{I_2 I_3} L_2 L_3 \tag{1}$$

$$\dot{L}_2 = \frac{I_3 - I_1}{I_1 I_3} L_1 L_3 \tag{2}$$

$$\dot{L}_3 = \frac{I_1 - I_2}{I_1 I_3} L_1 L_3. \tag{3}$$

The Euler equations can be written in a non-canonical form

$$\dot{L}_i = \sum_j J_{ij} \frac{\partial H}{\partial L_j} \tag{4}$$

where the Hamiltonian H is given by the total energy

$$H = \frac{1}{2} \sum L_i^2 / I_i \tag{5}$$

and the Poisson-tensor J reads as

$$J = \begin{pmatrix} 0 & -L_3 & L_2 \\ L_3 & 0 & -L_1 \\ -L_2 & L_1 & 0 \end{pmatrix}. \tag{6}$$

In case J is singular, further conserved quantities denoted as distinguished functions or Casimirs C can exist which are defined in general by

$$\sum_j J_{ij} \frac{\partial C}{\partial L_j} = 0. \tag{7}$$

The Casimirs of the Euler equations are arbitrary functions $C = C(L^2)$ of the length of the angular momentum vector $\boldsymbol{L}$.

The Euler equations can be written in Nambu representation that uses a Casimir, $G = L^2/2$, as a second conserved quantity besides H (Nambu 1973):

$$\dot{L}_i = \frac{\partial(L_i, G, H)}{\partial(L_1, L_2, L_3)} = \sum_{j,k} \varepsilon_{ijk} \frac{\partial G}{\partial L_j} \frac{\partial H}{\partial L_k}. \tag{8}$$

The Levi-Civita symbol ε_{ijk} could be denoted as the constant non-singular Poisson-tensor of rank 3.

The non-canonical operator J can be obtained by

$$J_{ij} = \frac{\partial(L_i, G, L_j)}{\partial(L_1, L_2, L_3)} \tag{9}$$

which relates Nambu mechanics to non-canonical Hamiltonian mechanics. As the roles of H and G are interchangeable, an alternative to the non-canonical representation is

$$\dot{L}_i = \sum_j \tilde{J}_{ij} \frac{\partial G}{\partial L_j} \tag{10}$$

which uses G as the generator of the time-evolution with the operator

$$\tilde{J}_{ij} = \frac{\partial(L_i, L_j, H)}{\partial(L_1, L_2, L_3)}. \tag{11}$$

In incompressible hydrodynamics enstrophy (2D) and helicity (3D) are known as integral conserved quantities besides energy. These two quantities which originate in the particle relabelling symmetry of the Lagrangian description are Casimir functionals of non-canonical Eulerian fluid mechanics. Therefore, it seems reasonable to write the hydrodynamical equations in a Nambu representation which involves total energy and the corresponding Casimir functional.

Incompressible inviscid fluid dynamics in 3D is governed by the vorticity equation

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = \boldsymbol{\xi} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{\xi} \tag{12}$$

and $\nabla \cdot \boldsymbol{u} = 0$, where $\boldsymbol{u}$ denotes the velocity and $\boldsymbol{\xi} = \nabla \times \boldsymbol{u}$ is the vorticity vector. The total energy

$$H = \frac{1}{2} \int \mathrm{d}^3 x \, \boldsymbol{u}^2 = -\frac{1}{2} \int \mathrm{d}^3 x \, \boldsymbol{\xi} \cdot \boldsymbol{A} \tag{13}$$

and the (total) helicity (Moffatt 1969)

$$h = \frac{1}{2} \int \mathrm{d}^3 x \, \boldsymbol{\xi} \cdot \boldsymbol{u} \tag{14}$$

are conserved (it is assumed that $\boldsymbol{u}$ vanishes at infinity). $\boldsymbol{A}$ is the vector potential, $\boldsymbol{u} = -\nabla \times \boldsymbol{A}$, with $\nabla \cdot \boldsymbol{A} = 0$.

The non-canonical form (Olver 1986) of the vorticity equation is

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = J(\boldsymbol{\xi}) \frac{\delta H}{\delta \boldsymbol{\xi}} \tag{15}$$

with the antisymmetric operator

$$J(\boldsymbol{\xi}) = -\nabla \times (\boldsymbol{\xi} \times \nabla \times (\cdot)) \,. \tag{16}$$

The derivative of the energy with respect to the vorticity is given by $\delta H / \delta \boldsymbol{\xi} = -\boldsymbol{A}$. Helicity is a Casimir functional, i.e. $J \delta h / \delta \boldsymbol{\xi} = 0$, because $\delta h / \delta \boldsymbol{\xi} = \boldsymbol{u}$.

We propose that the vorticity equation could be written as

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = K \left(\frac{\delta h}{\delta \boldsymbol{\xi}}, \frac{\delta H}{\delta \boldsymbol{\xi}} \right) \tag{17}$$

with

$$K(1,2) = -\nabla \times [(\nabla \times (1)) \times (\nabla \times (2))] \,. \tag{18}$$

K is a constant bilinear and antisymmetric operator which clearly yields (16). An arbitrary functional $F = F[\boldsymbol{\xi}]$ evolves according to

$$\frac{\partial F}{\partial t} = -\int \mathrm{d}^3 x \, (\nabla \times \frac{\delta F}{\delta \boldsymbol{\xi}}) \times (\nabla \times \frac{\delta h}{\delta \boldsymbol{\xi}}) \cdot (\nabla \times \frac{\delta H}{\delta \boldsymbol{\xi}}) = \{F, h, H\}_{3\mathrm{D}} . \tag{19}$$

In the last expression a generalized trilinear Poisson bracket has been introduced, which is antisymmetric in all its arguments. Helicity is no longer a hidden conserved quantity but enters the dynamics on the same level as the Hamiltonian. Obviously, (17) incorporates the non-canonical Hamiltonian representation and all applications of this theory can be performed starting from (17). For example, space translations are generated by the Kelvin momentum (Lamb 1932, p 152) according to the theorem of Noether

$$\boldsymbol{M} = \int \mathrm{d}^3 x \, \boldsymbol{u} = \frac{1}{2} \int \mathrm{d}^3 x \, \boldsymbol{r} \times \boldsymbol{\xi}. \tag{20}$$

Using $\delta \boldsymbol{M} / \delta \boldsymbol{\xi} = -\frac{1}{2} (\boldsymbol{r} \times \boldsymbol{I})$ with a 3×3 unit matrix $\boldsymbol{I}$, $\boldsymbol{M}$ determines

$$-\nabla F = \{F, h, \boldsymbol{M}\}_{3\mathrm{D}} . \tag{21}$$

In two dimensions, incompressible hydrodynamics is governed by the vorticity equation for $\zeta = \xi_z$

$$\frac{\partial \zeta}{\partial t} = -\boldsymbol{u} \cdot \nabla \zeta \tag{22}$$

together with $\nabla \cdot \boldsymbol{u} = 0$. In non-canonical Hamilton representation, this reads as

$$\frac{\partial \zeta}{\partial t} = J(\zeta)\frac{\delta H}{\delta \zeta} = -\mathcal{J}(\zeta, \frac{\delta H}{\delta \zeta}) \tag{23}$$

where $\mathcal{J}$ is the Jacobi-operator, $\mathcal{J}(a, b) = \partial_x a \partial_y b - \partial_y a \partial_x b$. The Hamiltonian is

$$H = \frac{1}{2}\int \mathrm{d}^2 x\, \boldsymbol{u}^2 = -\frac{1}{2}\int \mathrm{d}^2 x\, \zeta \psi \tag{24}$$

where ψ is the stream function for $\boldsymbol{u}$, $\boldsymbol{u} = \boldsymbol{k} \times \nabla\psi$ ($\boldsymbol{k}$ denotes the z-unit vector). The derivative of H with respect to ζ is $\delta H/\delta\zeta = -\psi$.

Casimir functionals are given by the integrals of arbitrary functions f of the vorticity

$$C = \int \mathrm{d}^2 x\, f(\zeta) \tag{25}$$

among them the most well known is enstrophy

$$\mathcal{E} = \frac{1}{2}\int \mathrm{d}^2 x\, \zeta^2. \tag{26}$$

The 2D vorticity equation can be expressed using the enstrophy in a way similar to (17)

$$\frac{\partial \zeta}{\partial t} = -\mathcal{J}\left(\frac{\delta \mathcal{E}}{\delta \zeta}, \frac{\delta H}{\delta \zeta}\right). \tag{27}$$

Here enstrophy could be replaced by any Casimir C (25) provided that $\mathcal{J}$ is replaced by $(f'')^{-1}\,\mathcal{J}$.

Time-evolution of an arbitrary functional $F = F[\zeta]$ in 2D reads as

$$\frac{\partial F}{\partial t} = -\int \mathrm{d}^2 x\, \frac{\delta F}{\partial \zeta}\mathcal{J}\left(\frac{\delta \mathcal{E}}{\delta \zeta}, \frac{\delta H}{\delta \zeta}\right) = \{F, \mathcal{E}, H\}_{2\mathrm{D}} \tag{28}$$

with a trilinear antisymmetric bracket. One could consider the operator K in (18) as a 3D analogue of the 2D Jacobi operator $\mathcal{J}$.

The Nambu brackets (19) and (28) satisfy the Jacobi identity when they are reduced to a Poisson bracket by keeping one argument, e.g. h, fixed

$$\{h, g, \{h, f, k\}\} + \{h, f, \{h, k, g\}\} + \{h, k, \{h, g, f\}\} = 0. \tag{29}$$

In summary, we have extended Nambu mechanics to incompressible hydrodynamics. The result is a generalization of the non-canonical Hamiltonian formulation, whereby the singular Poisson bracket is replaced by a non-singular multilinear antisymmetric bracket. In the new formulation, the Casimirs of the Hamiltonian theory (helicity in 3D and enstrophy in 2D) determine the dynamics on the same level as the Hamiltonian. The Nambu representation involves the two quantities which through their cascades determine the statistical behaviour of turbulent flows (for a recent reference to the helicity cascade, see Sanada (1993)). Work on compressible flows is in progress and will be published elsewhere.

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